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Another Green function for some birational maps of $\mathbb{P}^2$

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1 Introduction

Recently, several authors (, for example, [HP], [FS] and [U],) began to study the iteration theory of the rational maps of $\mathbb{P}^n$. The notion of the Green function (cf. §2) was introduced into this theory and played a decisive role there. This Green function also played the central role in the study of the Hénon maps (or more generally the finite composition of the generalized Hénon maps) of $\mathbb{P}^2$ or $\mathbb{C}^2$. (See, for example, [BS].)

Let $F=(F_1,\cdots,F_{n+1}):\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ be a holomorphic map defined by $(n+1)$ homogeneous polynomials $F_1,\cdots,F_{n+1}$ of degree $d\geq 2$ without common factor, and let $f:\mathbb{P}^n\cdots\to\mathbb{P}^n$ be the induced rational map. Denote by $\rho:\mathbb{C}^{n+1}\backslash\{0\}\to\mathbb{P}^n$ the usual projection map so that we have $\rho\circ F_{|\mathbb{C}^{n+1}\backslash\{0\}}=f\circ\rho$ outside the set $F^{-1}(\{0\})$ of $\mathbb{C}^{n+1}\setminus\{0\}$. The symbol $\cdots\to$ is used to emphasize that f is not necessary a holomophic map and may contain the points of indeterminacy.

Definition 1.1 *When the $(n+1)$ components of the k times iteration F^k have a common factor for some $k\geq 2$, we say that the degree lowering occurs for f.*

In the paper [FS], Fornaess and Sibony indicated that this kind of degree lowering phenomenon causes some difficulties in the study of the dynamics of rational maps of $\mathbb{P}^n$. In the case where the degree lowering occurs for f, it happens that the Green function is of no use for the study of iteration of f. (See §3 below.)

This note is an attempt to define another Green function which is useful in the iteration theory of rational maps for which the degree lowering occurs. The idea for defining another Green function, which will be explained in (2.2) is simple, but the proof of the convergence of the limit in (2.2) seems difficult. So, in this note, we only deal with a special example of rational maps. We will explain here the background of this example. In the note [N], we gave the list of the representatives of the birational polynomial quadratic maps of $\mathbb{P}^2$ under the conjugation by projective linear transformations $(PGL(3,\mathbb{C}))$ as the equivalence relation. In this note, we investigate the third family of the class B in the table in p.153 of [N], which is given by $\tilde{\varphi}:[z:w:t]\cdots\to[wt+\beta t^2:zw+\gamma t^2:t^2]$, where $(\beta,\gamma)\in\mathbb{C}^2$ and $[z:w:t]$ is the homogeneous coordinates of $\mathbb{P}^2$. Let us define $(b,c)\in\mathbb{C}^2$ by $\beta=c-b,\gamma=(1-c)b$, $f\in PGL(3,\mathbb{C})$ by $[z:w:t]\to[z-ct:w-bt:t]$, and the map φ by $\varphi=f\circ\tilde{\varphi}\circ f^{-1}$. Then, we have

$$\varphi:[z:w:t]\cdots\to[wt,zw+bzt+cwt:t^2]. \tag{1.1}$$

The family $\{\varphi_{b,c}\}$ is not the list of the representatives. In fact, it is easy to see that two maps $\varphi_{b,c}$ and $\varphi_{b',c'}$ are both conjugate to the map $\tilde{\varphi}_{\beta,\gamma}$ when $b' = 1 - c$ and $c' = 1 - b$. However, it seems that the formula (1.1) makes some calculation simpler than the formula of $\{\tilde{\varphi}_{\beta,\gamma}\}$. In this paper, φ always denote the map of (1.1) and ψ its inverse given by

$$\psi : [z : w : t] \cdots \to [(w - cz)t : (z + bt)z : (z + bt)t]. \tag{1.2}$$

The lifts to $\mathbb{C}^3$ of φ and ψ are always denoted by Φ and Ψ. By the coordinates (z, w, t) of $\mathbb{C}^3$, we have

$$\Phi : (z, w, t) \to (wt, zw + bzt + cwt, t^2) \tag{1.3}$$

and

$$\Psi : (z, w, t) \to ((w - cz)t, (z + bt)z, (z + bt)t). \tag{1.4}$$

Let us denote $(z_k, w_k, t_k) = \Phi^k(z, w, t)$ where Φ^k is the k times iteration of Φ. Then, we see that the three components z_2, w_2, t_2 of Φ^2 have the common factor t because $(z_2, w_2, t_2) = (w_1 t^3, (ww_1 + bwt^2 + cw_1 t)t, t^4)$. So, the degree of the induced rational map

$$\varphi^2 : [z : w : t] \cdots \to [w_1 t^2 : ww_1 + bwt^2 + cw_1 t : t^3] \tag{1.5}$$

of $\mathbb{P}^2$ is equal to 3 though the degree of φ is equal to 2.
Similary, we can see that the degree lowering occurs for the map ψ. It is desireable to define an another Green function for the map φ and ψ. However, we have succeeded so far only to prove the convergence of (2.2) for φ under a condition on (b, c) (Theorem 5.1). Here we remark that, for every $k \geq 1$, the points of indeterminacy of φ^k are always two points

$$I_1 = [0 : 1 : 0] \quad \text{and} \quad I_2 = [1 : 0 : 0], \tag{1.6}$$

whereas the number of the points of indeterminacy of ψ^k for general (b, c) increases as k grows.
In §2, we give a brief review of the Green function of the rational maps of $\mathbb{P}^n$ and an idea to define another Green function. In §3, we study the maps φ and ψ when $(b, c) = (0, 0)$. In this very special case, we can describe completely the k times iterates. So we can easily define another Green functions for φ and ψ. In §§4–5, the iteration of φ will be investigated. In §4, we state two Propositions 4.5 and 4.6, for which we need not pose any assumption on (b, c). In §5, under a condition on (b, c), we prove the convergence of the limit in (2.2).

2 Green function

Theorem 2.1 ([HP], [FS], [U]) *Let $F = (F_1, \cdots, F_{n+1}) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a holomorphic map defined by $(n + 1)$ homogeneous polynomials $F_1, \cdots, F_{n+1}$ of degree $d \geq 2$ without common factor, and let $f : \mathbb{P}^n \cdots \to \mathbb{P}^n$ be the induced rational map. Let*

$$H(p) := \lim_{k \to \infty} \frac{1}{d^k} \log |F^k(p)|, \tag{2.1}$$

where the norm $|*|$ *of* $|F^k(p)|$ *is, say, the maximum norm.*
(1) $H : \mathbb{C}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ *is a plurisubharmonic function or is identically equal to* $-\infty$.
(2) $H(\lambda p) = H(p) + \log|\lambda| \quad (\lambda \in \mathbb{C}^*)$.
(3) $H(F(p)) = d \cdot H(p)$.

A rational map $f : \mathbb{P}^n \cdots \to \mathbb{P}^n$ is holomorphic if and only if $F^{-1}(\{0\}) = \{0\}$. When f is holomorphic the Green function H reflects the dynamical properties of f, as is illustrated by the following theorem.

Theorem 2.2 ([HP], [FS], [U]) *Suppose that* $f : \mathbb{P}^n \to \mathbb{P}^n$ *is a holomorphic map and let*

$$\mathcal{H} = \{p \in \mathbb{C}^{n+1} \setminus \{0\}; H \text{ is pluriharmonic in a neighborhood of } p\},$$

and $\Omega = \rho(\mathcal{H})$. *Then,* Ω *coincides with the Fatou set of* f.

When $n = 2$ and f is the Hénon map (or the finite composition of the generalized Hénon maps), the Green function H plays an important role. For simplicity, we consider the Hénon map $f : \mathbb{P}^2 \cdots \to \mathbb{P}^2$ given by $[z : w : t] \cdots \to [wt : w^2 - azt + ct^2 : t^2]$ with $(a, c) \in \mathbb{C}^* \times \mathbb{C}$. Usually the Hénon map is dealt with as a holomorphic map f of $\mathbb{C}^2$ by restricting f to $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{t = 0\}$. Let us denote by $(x = \frac{z}{t}, y = \frac{w}{t})$ the holomorphic coordinates of $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{t = 0\}$ so that we have $f(x, y) = (y, y^2 - ax + c)$. Then $h(x, y) := H(x, y, 1)$ on $\mathbb{C}^2$ is the Green function of Bedford and Smillie [BS] and other authors. The proof of Theorem 2.1 ([FS], [U]) is very elegant and short. Especially, when $n = 2$ and f on $\mathbb{P}^2$ or on $\mathbb{C}^2$ is the Hénon map, considering the lifts $F : \mathbb{C}^3 \to \mathbb{C}^3$ of f and applying Theorem 2.1 give the shortest proof for the convergence of $h(x, y) = \lim_{k\to\infty} \log^+|f^k(x, y)|$.
We remark that the degree lowering does not occur for the Hénon map.
Now, we turn our attention to our map φ of (1.1).

Definition 2.3 (Fibonacci *sequence*) *Define a sequence* $\{\nu_k\}$ *by the recursion relation*

$$\nu_{k+2} = \nu_{k+1} + \nu_k, \text{ with } \nu_1 = 1,\ \nu_2 = 1.$$

By the notation $\omega = \frac{1+\sqrt{5}}{2}$ *and* $\omega_1 = \frac{-1}{\omega}$, *we have* $\nu_k = \frac{\omega^k - \omega_1^k}{\sqrt{5}} \quad (k \geq 1)$.

Proposition 2.4 *Let* $p = (z, w, t) \in \mathbb{C}^3$ *and* $p_k = (z_k, w_k, t_k) = \Phi^k(p)$. *Then, the common factor* $\Delta_k(p)$ *of* $\Phi^k(p)$ *is* $\Delta_k(p) = t^{2^k - \nu_{k+2}}$. *Therefore, letting* $\hat{p} = (\hat{z}_k, \hat{w}_k, \hat{t}_k) := \frac{p_k}{\Delta_k(p)}$, *the degree of the map* $\varphi^k : [z : w : t] \cdots \to [\hat{z}_k : \hat{w}_k : \hat{t}_k]$, *which means the common degree of the components of* $\hat{p}_k$, *is equal to* ν_{k+2}.

Proof. Let a_k and b_k be the multiplicities of the factor t contained in z_k, w_k. First, by induction on k, we will show that $a_k = b_{k-1} + 2^{k-1} < 2^k, b_k = a_{k-1} + b_{k-1} < 2^k (k \geq 2)$.
By (1.3),we have $a_1 = 1 < 2, b_1 = 0 < 2$. By $t_{k-1} = t^{2^{k-1}}$ and

$$z_k = w_{k-1}t^{2^{k-1}}, w_k = z_{k-1}w_{k-1} + bz_{k-1}t^{2^{k-1}} + cw_{k-1}t^{2^{k-1}},$$

the induction hypothesis on $k-1$ yields the assertion on k. Hence, letting $d_k = 2^k - b_k$, $\{d_k\}$ satisfy $d_{k+2} = d_{k+1} + d_k, d_1 = 2, d_2 = 3$. Refering to Definition 2.3, we have $d_k = \nu_{k+2}$, therefore $a_k = 2^k - \nu_{k+1}, b_k = 2^k - \nu_{k+2}$. □

The idea to define another Green function $G(p)$ on $\mathbb{C}^3$ for the map φ is simple and seems reasonable. Using the notation of Proposition 2.4, we consider

$$G_k(p) := \frac{1}{\nu_{k+2}} \log |\frac{\Phi^k(p)}{\Delta_k(p)}|, \quad G(p) := \lim_{k\to\infty} G_k(p), \tag{2.2}$$

where $|\cdot|$ is the maximum norm of $\mathbb{C}^3$. Of course, this idea may be applicable for the general rational maps f of $\mathbb{P}^2$ or $\mathbb{P}^n$. Denoting the lift of f by F, the common factor of $F^k(p)$ by $\Delta_k(p)$, the degree of $F^k(p)/\Delta_k(p)$ by d_k, what we want to consider is $lim_{k\to\infty} \frac{1}{d_k} \log |F^k(p)/\Delta_k(p)|$. But, contrary to Theorem 2.1, the proof of the convergence of this limit seems not so easy. In the final section, we will prove the convergence of (2.2) for φ when $|b-c|<1$.

3 φ and ψ when b=c=0

When $b = c = 0$, we can describe φ^k, ψ^k explicitly. Therefore, we can define another Green function for both φ and ψ. In this §3, φ, ψ, Φ and Ψ always denote (1.1),(1.2), (1.3) and (1.4) with $b = c = 0$.

Proposition 3.1 *Suppose $b = c = 0$. We set $\nu_0 = 0$.*
(a) For $k \geq 1$, $p_k = (z_k, w_k, t_k) = \Phi^k(p) = (z^{\nu_{k-1}} w^{\nu_k} t^{2^k - \nu_{k+1}}, z^{\nu_k} w^{\nu_{k+1}} t^{2^k-\nu_{k+2}}, t^{2^k})$.
(b) The Green function of (2.1) is $H(p) = \log|t|$.
(c) The Fatou set of f consists of two components

$$\Omega_1 = \{[z:w:t] \in \mathbb{P}^2; |z||w|^{\omega} > |t|^{\omega^2}\}, \text{ and } \Omega_2 = \{[z:w:t] \in \mathbb{P}^2; |z||w|^{\omega} < |t|^{\omega^2}\}$$

and we have $\varphi(\Omega_1) = (\Omega_1)$ and $\varphi(\Omega_2) = (\Omega_2)$. Here we remark that $\omega^2 = 1 + \omega$.
(d) The expression of (2.2) converges uniformly on every compact on $\mathbb{C}^3$ to

$$G(p) = \begin{cases} \dfrac{1}{\omega^2} \log|z| + \dfrac{1}{\omega} \log|w| & (\text{when } |z||w|^{\omega} \geq |t|^{\omega^2}) \\ \log|t| & (\text{when } |z||w|^{\omega} \leq |t|^{\omega^2}). \end{cases}$$

(e) The function $G(p)$ has the following properties:
  (1) $G: \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic.
  (2) $G(\lambda p) = G(p) + \log|\lambda|$ $(\lambda \in \mathbb{C}^*)$.
  (3) $G(\Phi(p)) = \omega G(p) + \frac{1}{\omega^2}\log|t|$.

Proof. By Proposition 2.4, we can put $z_k = z^{\alpha_k} w^{\beta_k} t^{2^k - \nu_{k+1}}, w_k = z^{\gamma_k} w^{\delta_k} t^{2^k - \nu_{k+2}}$.
Then, by $z_k = w_{k-1}t_{k-1}, w_k = z_{k-1}w_{k-1}$, we have

$$\alpha_k = \gamma_{k-1}, \beta_k = \delta_{k-1}, \gamma_k = \alpha_{k-1} + \gamma_{k-1}, \text{ and } \delta_k = \beta_{k-1} + \delta_{k-1},$$

from which it holds that

$$\gamma_{k+2} = \gamma_{k+1} + \gamma_k, \gamma_1 = 1, \gamma_2 = 1; \delta_{k+2} = \delta_{k+1} + \delta_k, \delta_1 = 1, \delta_2 = 2.$$

By Definition 2.3, we have $\gamma_k = \nu_k, \alpha_k = \nu_{k-1}, \delta_k = \nu_{k+1}, \beta_k = \nu_k$, which imply the assertion (a). Since $\omega = 1.6\cdots < 2$, (b) follows immediately from (a). Since $\frac{\Phi_k(p)}{\Delta_k(p)} = (\hat{z}_k, \hat{w}_k, \hat{t}_k) = (z^{\nu_{k-1}}w^{\nu_k}t^{\nu_k}, z^{\nu_k}w^{\nu_{k+1}}, t^{\nu_{k+2}})$, we have $G(p) = \frac{1}{\omega^3}\log\max(|z||w|^\omega|t|^\omega, |z|^\omega|w|^{\omega^2}, |t|^{\omega^3})$. When $|z||w|^\omega|t|^{-\omega^2} \le 1$, it holds $|z|^\omega|w|^{\omega^2} \le |z||w|^\omega|t|^\omega \le |t|^{\omega^3}$, hence $G(p) = \log|t|$.
On the other hand, when $|z||w|^\omega|t|^{-\omega^2} \ge 1$, it holds $|t|^{\omega^3} \le |z||w|^\omega|t|^\omega \le |z|^\omega|w|^{\omega^2}$, hence $G(p) = \frac{1}{\omega^2}\log|z| + \frac{1}{\omega}\log|w|$, which is the assertion (d). The assertions (1), (2) and (3) of (e) are immediately shown by (d).
Finally, we will prove (c). Note that all the points of indeterminacy of φ in (1.6) satisfy $\{I_1, I_2\} \cap (\Omega_1 \cup \Omega_2) = \emptyset$. By direct calculation, we can see $\varphi(\Omega_1) = \Omega_1$ and $\varphi(\Omega_2) = \Omega_2$. We will show that $\{\varphi^k\}$ converges to the constant map $I_1 = [0 : 1 : 0]$ uniformly on every compact of $\Omega_1 = \{|z||w|^\omega > |t|^{\omega^2}\}$. In fact, since $\Omega_1 \subset \mathbb{P}^2\backslash(\{w = 0\} \cup \{z = 0\})$ we investigate in the coordinates $u = \frac{z}{w}, v = \frac{t}{w}$. Then, we have $\Omega_1 = \{|u| > |v|^{\omega^2}\}$ and $\varphi^k(u, v) = (u^{-\nu_{k-2}}v^{\nu_k}, u^{-\nu_k}v^{\nu_{k+2}})$. Therefore, when $(u, v) \in \Omega_1$, applying $\lim_{k\to\infty}\frac{\nu_{k+2}}{\nu_k} = \omega^2$, we see that $|u|^{-\nu_{k-2}}|v|^{\nu_k} \le (|v|^{\omega^2}|u|^{-1}|v|^{(\nu_k/\nu_{k-2}-\omega_1^2)})^{\nu_{k-2}} \to 0$.
Similarly, we can prove that $\{\varphi^k\}$ converge to the constant map $[0 : 0 : 1]$ uniformly on every compact in $\Omega_2 = \{|z||w|^\omega < |t|^{\omega^2}\}$. □

Propositon 3.1, (b), shows that it happens that $H(p)$ does not reflect the dynamics of φ.

Proposition 3.2 *(a) Let $k \ge 2$ be an even integer and let*
$A_k = \frac{2^{k+1}+(\omega^{k+2}+\omega_1^{k+2})}{5}$, $B_k = \frac{2^k-(\omega^{k+1}+\omega_1^{k+1})}{5}$, $C_k = \frac{2^{k+1}-(\omega^k+\omega_1^k)}{5}$,
$D_k = \frac{2^{k+1}-2(\omega^{k+1}+\omega_1^{k+1})}{5}$, $E_k = \frac{2^k+2(\omega^k+\omega_1^k)}{5}$, $F_k = \frac{2^{k+1}+2(\omega^{k-1}+\omega_1^{k-1})}{5}$,
$G_k = \frac{2^{k+1}-(\omega^k+\omega_1^k)}{5}$, $H_k = \frac{2^k+(\omega^{k-1}+\omega_1^{k-1})}{5}$, $I_k = \frac{2^{k+1}+(\omega^{k-2}+\omega_1^{k-2})}{5}$.
Then, $p_k = (z_k, w_k, z_k) = \Psi^k(p)$ of Ψ in (1.4) satisfy

$$\Psi^k(p) = (z^{A_k}w^{B_k}t^{C_k}, z^{D_k}w^{E_k}t^{F_k}, z^{G_k}w^{H_k}t^{I_k}). \tag{3.1}$$

The common factor $\Delta_k(p)$ of three components of $\Psi^k(p)$ is

$$\Delta_k(p) = z^{D_k}w^{B_k}t^{C_k}, \text{ where we have } D_k + B_k + C_k = 2^k - \nu_{k+2}.$$

(b) The Green function of (2.1) is $H(p) = \frac{1}{5}\log|z|^2|w||t|^2$.
(c) The Fatou set of ψ consists of two components

$$\Omega_1 = \{[z : w : t]; |z|^{\omega^2} > |w|^\omega|t|\} \text{ and } \Omega_2 = \{[z : w : t]; |z|^{\omega^2} < |w|^\omega|t|\}$$

satisfying $\psi(\Omega_1) = \Omega_2$ and $\psi(\Omega_2) = \Omega_1$.
(d) For $G_k(p) := \frac{1}{\nu_{k+2}}\log|\frac{p_k}{\Delta_k(p)}|$, $G(p) = \lim_{k\to\infty} G_k(p)$ converges uniformly on every compact in $\mathbb{C}^3$ to $G(p) = \begin{cases} \log|z| & (\text{when } |z|^{\omega^2} \ge |w|^\omega|t|) \\ \frac{1}{\omega}\log|w| + \frac{1}{\omega^2}\log|t| & (\text{when } |z|^{\omega^2} \le |w|^\omega|t|). \end{cases}$

(e) *This $G(p)$ satifies the following properties.*

(1) $G(p) : \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ *is plurisubharmonic on* $\mathbb{C}^3$.

(2) $G(\lambda p) = G(p) + \log|\lambda| \quad (\lambda \in \mathbb{C}^*)$.

(3) $G(\Psi(p)) = \omega G(p) + \dfrac{1}{\omega^2}\log|t|$.

Proof. We can prove (3.1) by induction on even integer k. Since we can see easily that $D_k \leq G_k \leq A_k$, $B_k \leq H_k \leq E_k$ and $C_k \leq I_k \leq F_k$, the assertion on $\Delta_k(p)$ follows.
Now, it is easy to deduce the expression of p_k for odd integers. The assertion (b) can be seen by the expressions for both even and odd integers.
The points of indeterminacy of ψ are $J_1 = [0:1:0]$ and $J_2 = [0:0:1]$ and $(\Omega_1 \cup \Omega_2) \cap \{J_1, J_2\} = \emptyset$. So, we can see by direct calculation that $\psi(\Omega_1) = \Omega_2$ and $\psi(\Omega_2) = \Omega_1$.
For (d), first, we give the proof when $k \to \infty$ with even integers k. By using (3.1), we have $\frac{|p_k|}{|\Delta_k(p)|} = (|z|^{A_k - D_k}, |w|^{E_k - B_k}|t|^{F_k - C_k}, |z|^{G_k - D_k}|w|^{H_k - B_k}|t|^{I_k - C_k})$.
Then, we have $G(p) = \frac{1}{\omega^3}\log\max(|z|^{\omega^3}, |w|^{\omega^2}|t|^{\omega}, |z|^{\omega}|w|^{\omega}|t|)$, since it hold $\frac{A_k - D_k}{\nu_{k+2}} \to 1$ $(k \to \infty)$ and so on. Since $|z|^{\omega^3} \geq |z|^{\omega}|w|^{\omega}|t| \geq |w|^{\omega^2}|t|^{\omega}$ when $|z|^{\omega^2}|w|^{-\omega} \geq |t|$ and $|z|^{\omega^3} \leq |z|^{\omega}|w|^{\omega}|t| \leq |w|^{\omega^2}|t|^{\omega}$ when $|z|^{\omega^2}|w|^{-\omega} \leq |t|$, we have the expression of (d). We can prove the same expression (d) when letting $k \to \infty$ with odd integers.
All the assertions of (e) are clear by (d).
Finally, we will prove the first part of (c). We will show that the sequence $\{\psi^k; k \text{ even}\}$ converges to the constant map $[1:0:0]$ uniformly on every compact in Ω_1. Since $\Omega_1 \subset \mathbb{P}^2 \backslash \{z = 0\}$, we investigate in the coordinates $f = \frac{w}{z}, g = \frac{t}{z}$. Then, $\Omega_1 = \{|f|^{\omega}|g| < 1\}$ and, for even k, $\psi^k(f,g) = (f^{\nu_{k+1}}g^{\nu_k}, f^{\nu_k}g^{\nu_{k-1}})$. Here we used $E_k - B_k = \nu_{k+1}$, $F_k - C_k = \nu_k$, $H_k - B_k = \nu_k$ and $I_k - C_k = \nu_{k-1}$. Then, for $(f,g) \in \Omega_1$, we have $|f|^{\nu_{k+1}}|g|^{\nu_k} = ((|f|^{\omega}|g|)|f|^{\nu_{k+1}/\nu_k - \omega})^{\nu_k} \to 0$.
Similarly, we can prove that the sequence $\{\psi^k; k \text{ even}\}$ converges to the constant map $[0:1:0]$ uniformly on every compact in Ω_2. □

4 Two propositions on φ

In this section, we do not pose any assumption on (b, c).
First, we will consider $\varphi_{|\mathbb{C}^2} : (x,y) \to (y, xy + bx + cy)$ by the coordinates $x = \frac{z}{t}, y = \frac{w}{t}$ of $\mathbb{C}^2 = (\mathbb{P}^2 - \{t = 0\})$ and let us denote $(x_k, y_k) = \varphi^k(x,y)$.

Lemma 4.1 *Take $A > 0$ with*

$$A \geq 2 + |b| + |c|, A \geq 2|c|, A \geq 2|b| \tag{4.1}$$

and let $W_A = \{|y| \geq A, |x| \geq A\}$. Then we have $\varphi(W_A) \subset W_A$ and $\varphi^2(W_A) \subset W_{2A}$.

Proof. Let $(x, y) \in W_A$. Then $|x_1| = |y| \geq A$, and

$$|y_1| \geq |x||y| - |b||x| - |c||y| = (\frac{1}{2}|x| - |c|)|y| + (\frac{1}{2}|y| - |b|)|x|$$

$$\geq (\frac{A}{2} - |c|)A + (\frac{A}{2} - |b|)A = (A - |b| - |c|)A \geq 2A.$$

So, $\varphi(W_A) \subset W_A$. Furthermore, for $(x_1, y_1) \in W_A$, the above calculation implies $|x_2| = |y_1| \geq 2A$ and $|y_2| \geq 2A$. □

In the following Lemmas 4.2 and 4.3, we fix δ with $0 < \delta < 1$, and $A_1 > 0$ satisfying

$$A_1 \geq 2 + |b| + |c|, A_1 \geq 2|c|, A_1 \geq 2|b|, \delta \geq \frac{|b| + |c|}{A_1 - |b| - |c|} \text{ and } \delta \geq \frac{|b| + |c|}{A_1}. \tag{4.2}$$

Lemma 4.2 *For $(x, y) \in W_{A_1}$, we have $|y| \leq |x_1| \leq |y|, (1+\delta)^{-1}|xy| \leq |y_1| \leq (1+\delta)|xy|$.*

Proof. The inequalities for $|y_1|$ can be seen by

$$|y_1| \geq |x||y| - \frac{|b|}{|y|}|x||y| - \frac{|c|}{|x|}|x||y| \geq |x||y| - \frac{|b|}{A_1}|x||y| - \frac{|c|}{A_1}|x||y|$$

$$= \frac{A_1 - |b| - |c|}{A_1}|x||y| \geq \frac{1}{1+\delta}|x||y|,$$

and

$$|y_1| \leq |x||y| + \frac{|b|}{|y|}|x||y| + \frac{|c|}{|x|}|x||y| \leq |x||y| + \frac{|b|}{A_1}|x||y| + \frac{|c|}{A_1}|x||y|$$

$$= \frac{A_1 + |b| + |c|}{A_1}|x||y| \leq (1+\delta)|x||y|. \qquad \square$$

Lemma 4.3 *For $(x, y) \in W_{A_1}$, we have*

$$(1+\delta)^{-(\nu_{k+1}-1)}|x|^{\nu_{k-1}}|y|^{\nu_k} \leq |x_k| \leq (1+\delta)^{\nu_{k+1}-1}|x|^{\nu_{k-1}}|y|^{\nu_k},$$

$$(1+\delta)^{-(\nu_{k+2}-1)}|x|^{\nu_k}|y|^{\nu_{k+1}} \leq |y_k| \leq (1+\delta)^{\nu_{k+2}-1}|x|^{\nu_k}|y|^{\nu_{k+1}}.$$

Proof. Let us define $\{N_k\}$ by $N_{k+2} = (1+\delta)N_{k+1}N_k$; $N_1 = (1+\delta), N_2 = (1+\delta)^2$, and let $M_{k+1} = N_k$ $(M_1 = 1)$. Let $\nu_0 = 0$ and let ν_k be as in Definition 2.3. Then, for $k \geq 1$ *and* $(x, y) \in W_{A_1}$, we have

$$M_k^{-1}|x|^{\nu_{k-1}}|y|^{\nu_k} \leq |x_k| \leq M_k|x|^{\nu_{k-1}}|y|^{\nu_k}, N_k^{-1}|x|^{\nu_k}|y|^{\nu_{k+1}} \leq |y_k| \leq N_k|x|^{\nu_k}|y|^{\nu_{k+1}}. \tag{4.3}$$

In fact, by Lemma 4.2, these inequalities hold for $k = 1$. The general case follows by induction on k. Let us define a sequence $\{s_k\}$ by the recursion relation $s_{k+2} = s_{k+1} + s_k + 1$ with $s_1 = 1, s_2 = 2$. Then, by Definition 2.3, it holds $s_k = \nu_{k+2} - 1$. Now (4.3) are the desired inequalities of the present lemma because $\log N_k = s_k \log(1+\delta)$. □

Definition 4.4 *For $q = (x, y) \in W_A$, and $G_k(p)$ of (2.2), let $g_k(x, y) = G_k(x, y, 1)$. Letting $g_k^{(1)}(q) = \frac{1}{\nu_{k+2}} \log|x_k|$ and $g_k^{(2)}(q) = \frac{1}{\nu_{k+2}} \log|y_k|$, it follows $g_k(q) = \max(g_k^{(1)}(q), g_k^{(2)}(q), 0)$.*

Proposition 4.5 *For A with (4.1), $\{g_k(q)\}$ converges uniformly on every compact in W_A.*

Proof. Fix $U \subset\subset W_A$ and $\varepsilon > 0$. Taking the boundedness of the sequence $\{\frac{\nu_{k+1}}{\nu_{k+2}}\}$ into consideration, we take $0 < \delta < 1$ such that, for all $k, l \in \mathbb{N}$ with $k > l$, we have

$$(\frac{\nu_{k+1}-1}{\nu_{k+2}} + \frac{\nu_{l+1}-1}{\nu_{l+2}})\log(1+\delta) < \varepsilon,\ (\frac{\nu_{k+2}-1}{\nu_{k+2}} + \frac{\nu_{l+2}-1}{\nu_{l+2}})\log(1+\delta) < \varepsilon. \tag{4.4}$$

We take $A_1 > 0$ satisfying (4.2). We take and fix a positive integer m with $\varphi^m(U) \subset\subset W_{A_1}$. This is possible by Lemma 4.1. For $q = (x, y) \in U$, put $(u, v) = q_m = \varphi^m(q) \in \varphi^m(U) \subset\subset W_{A_1}$. Note that, for $i = 1$ and $i = 2$, we have $g^{(i)}_{m+k}(q) = \frac{\nu_{k+2}}{\nu_{m+k+2}} g^{(i)}_k(q_m)$.
By Lemma 4.3 applied for $(u, v) \in W_{A_1}$ and by the boundedness of $\{\frac{\nu_{k-1}}{\nu_{k+2}}\}$, $\{\frac{\nu_k}{\nu_{k+2}}\}$, $\{\frac{\nu_{k+1}}{\nu_{k+2}}\}$, the sequence $\{g^{(i)}_k(q);\ k\}$ is bounded on $\varphi^m(U)$. Take $B > 0$ such that $|g^{(i)}_k(q_m)| \le B$ on U for $i = 1, 2$ and $k \in \mathbb{N}$.
Take $K > 0$ such that, for all $(k, l) \in \mathbb{N}^2$ with $k > l \ge K$ and for all $(u, v) \in \varphi^m(U)$,

$$|(\frac{\nu_{k-1}}{\nu_{k+2}} - \frac{\nu_{l-1}}{\nu_{l+2}})\log|u|| < \varepsilon, \qquad |(\frac{\nu_k}{\nu_{k+2}} - \frac{\nu_l}{\nu_{l+2}})\log|v|| < \varepsilon,$$

$$|(\frac{\nu_k}{\nu_{k+2}} - \frac{\nu_l}{\nu_{l+2}})\log|u|| < \varepsilon, \qquad |(\frac{\nu_{k+1}}{\nu_{k+2}} - \frac{\nu_{l+1}}{\nu_{l+2}})\log|v|| < \varepsilon. \tag{4.5}$$

This is possible since $\{\frac{\nu_{k-1}}{\nu_{k+2}}\}$, $\{\frac{\nu_k}{\nu_{k+2}}\}$ and $\{\frac{\nu_{k+1}}{\nu_{k+2}}\}$ are convergent.
Now by Lemma 4.3 applied for (u, v), (4.4) and (4.5), we have, for $q \in U$ and $k > l \ge K$,

$$|g^{(i)}_k(q_m) - g^{(i)}_l(q_m)| < 3\varepsilon. \tag{4.6}$$

Since $\{\frac{\nu_{k+2}}{\nu_{m+k+2}}\}$ is convergent, we can take $K_1 > K$ such that, for $k > l \ge K_1$,

$$B|\frac{\nu_{k+2}}{\nu_{m+k+2}} - \frac{\nu_{l+2}}{\nu_{m+l+2}}| < \varepsilon. \tag{4.7}$$

Then, by (4.6) and (4.7), for $q \in U$ and $k > l \ge K_1$, we have

$$|g^{(i)}_{k+m}(q) - g^{(i)}_{l+m}(q)| = |\frac{\nu_{k+2}}{\nu_{k+m+2}} g^{(i)}_k(q_m) - \frac{\nu_{l+2}}{\nu_{l+m+2}} g^{(i)}_l(q_m)|$$

$$\le \frac{\nu_{k+2}}{\nu_{k+m+2}}|g^{(i)}_k(q_m) - g^{(i)}_l(q_m)| + |\frac{\nu_{k+2}}{\nu_{k+m+2}} - \frac{\nu_{l+2}}{\nu_{l+m+2}}||g^{(i)}_l(q_m)| < 3\frac{\nu_{k+2}}{\nu_{k+m+2}}\varepsilon + \varepsilon \le 4\varepsilon.$$

Hence, for $q \in U$ and $k > l \ge (K_1 + m)$, we have $|g_k(q) - g_l(q)| \le 4\varepsilon$. □

In the next Proposition 4.6, we study the iteration of φ around two points of indeterminacy $I_1 = [0 : 1 : 0]$ and $I_2 = [1 : 0 : 0]$. For $\mu > 0$, let

$$\Lambda_1(\mu) = \{|z| \le \mu|w|,\ |t| \le \mu|w|\},\ \Lambda_2(\mu) = \{|w| \le \mu|z|,\ |t| \le \mu|z|\} \subset \mathbb{P}^2. \tag{4.8}$$

Because of Proposition 2.4, we can see that

$$(\hat{z}_{k+1}, \hat{w}_{k+1}, \hat{t}_{k+1}) = \hat{t}_k^{\,-\nu_k/\nu_{k+2}}(\hat{w}_k\hat{t}_k,\ \hat{z}_k\hat{w}_k + b\hat{z}_k\hat{t}_k + c\hat{w}_k\hat{t}_k,\ \hat{t}_k^2). \tag{4.9}$$

Proposition 4.6 *Let* $0 < \mu \leq 1$ *and* $\gamma = \max(|b|, |c|, 1)$.
(1) *For* $(z, w, t) \in \rho^{-1}(\Lambda_1(\mu))$, *we have*

$$|\hat{z}_k| \leq e^{(1+2\gamma)^2 \nu_{k-1}} \mu^{\nu_{k+1}} |w|^{\nu_{k+2}}, \ |\hat{w}_k| \leq e^{(1+2\gamma)^2 \nu_k} \mu^{\nu_k} |w|^{\nu_{k+2}}, \ |\hat{t}_k| \leq \mu^{\nu_{k+2}} |w|^{\nu_{k+2}}.$$

(2) *For* $(z, w, t) \in \rho^{-1}(\Lambda_2(\mu))$, *we have*

$$|\hat{z}_k| \leq e^{(1+2\gamma)^2 \nu_k} \mu^{2\nu_k} |z|^{\nu_{k+2}}, \ |\hat{w}_k| \leq e^{(1+2\gamma)^2 \nu_{k+1}} \mu^{\nu_{k+1}} |z|^{\nu_{k+2}}, \ |\hat{t}_k| \leq \mu^{\nu_{k+2}} |z|^{\nu_{k+2}}.$$

Proof. (1) Let $L_k = e^{(1+2\gamma)^2 \nu_k}$ and $B_k + \gamma \mu^{\nu_{k+1}} = L_k$ for $k \geq 1$. We will prove, for $k \geq 1$,

$$|\hat{z}_k| \leq B_{k-1} \mu^{\nu_{k+1}}, \ |\hat{w}_k| \leq B_k \mu^{\nu_k}, \ |\hat{t}_k| = \mu^{\nu_{k+2}}, \tag{4.10}$$

when $|z| \leq \mu$, $|w| = 1$, $|t| = \mu$ (Set $B_0 = 1$.). Then, by the maximum principle of the plurisubharmonic functions $|\hat{z}_k|$, $|\hat{w}_k|$ and $|\hat{t}_k|$, the estimates (4.10) hold when $|z| \leq \mu$, $|w| = 1$ and $|t| \leq \mu$, which implies the assertion of Proposition.
We will proceed by induction on k. When $k = 1$, we have

$$|z_1| \leq \mu, \ |w_1| \leq (1 + \mu|b| + |c|)\mu \leq B_1 \mu,$$

and when $k = 2$,

$$|\hat{z}_2| \leq \mu^{-1}(1 + \mu|b| + |c|)\mu^3 \leq B_1 \mu^2,$$

$$|\hat{w}_2| \leq \mu^{-1}\{(1 + \mu|b| + |c|)\mu^2 + |b|\mu^3 + |c|(1 + \mu|b| + |c|)\mu^3\} \leq B_2 \mu.$$

Assume that inequalities (4.10) hold. Then, by (4.9),

$$\begin{aligned}
|\hat{z}_{k+1}| &\leq \mu^{-\nu_k} |\hat{w}_k| \mu^{\nu_{k+2}} \leq B_k \mu^{\nu_{k+2}}, \\
|\hat{w}_{k+1}| &\leq \mu^{-\nu_k} \{B_{k-1} B_k \mu^{\nu_{k+2}} + |b| B_{k-1} \mu^{\nu_{k+1}+\nu_{k+2}} + |c| B_k \mu^{\nu_k+\nu_{k+2}}\} \\
&\leq \{(B_{k-1} + \gamma\mu^{\nu_k})(B_k + \gamma\mu^{\nu_{k+1}}) - \gamma^2 \mu^{\nu_{k+2}}\} \mu^{\nu_{k+1}} \\
&\leq \{(B_{k-1} + \gamma\mu^{\nu_k})(B_k + \gamma\mu^{\nu_{k+1}}) - \gamma \mu^{\nu_{k+2}}\} \mu^{\nu_{k+1}} = B_{k+1} \mu^{\nu_{k+1}}.
\end{aligned}$$

(2) Let $L_k = e^{(1+2\gamma)^2 \nu_k}$ and $B_k + \gamma \mu^{\nu_k} = L_k$ for $k \geq 1$. We will prove

$$|\hat{z}_k| \leq B_{k-1} \mu^{2\nu_k}, \ |\hat{w}_k| \leq B_k \mu^{\nu_{k+1}}, \ |\hat{t}_k| = \mu^{\nu_{k+2}}, \tag{4.11}$$

when $|z| = 1$, $|w| \leq \mu$ and $|t| = \mu$ and $k \geq 1$ (Set $B_0 = 1$.). Then, by the maximum principle of the plurisubharmonic functions $|\hat{z}_k|$, $|\hat{w}_k|$ and $|\hat{t}_k|$, the estimates (4.11) hold when $|z| = 1$, $|w| \leq \mu$ and $|t| \leq \mu$, which implies the assertion of Proposition.
We will proceed by induction on k. When $k = 1$,

$$|z_1| \leq \mu^2, |w_1| \leq (1 + |b| + \mu|c|)\mu \leq B_1 \mu.$$

When $k = 2$,

$$|\hat{z}_2| \leq \mu^{-1}(1 + |b| + \mu|c|)\mu^3 \leq B_1 \mu^2,$$

$$|\hat{w}_2| \leq \mu^{-1}\{(1+|b|+\mu|c|)\mu^3 + |b|\mu^4 + |c|(1+|b|+\mu|c|)\mu^3\} \leq B_2\mu^2.$$

Assume that the estimates (4.11) hold. Then, by (4.9),

$$\begin{aligned}
|\hat{z}_{k+1}| &\leq \mu^{-\nu_k} B_k \mu^{\nu_{k+1}} \mu^{\nu_{k+2}} = B_k \mu^{2\nu_{k+1}}, \\
|\hat{w}_{k+1}| &\leq \mu^{-\nu_k}\{B_{k-1}B_k\mu^{2\nu_k+\nu_{k+1}} + |b|B_{k-1}\mu^{2\nu_k+\nu_{k+2}} + |c|B_k\mu^{\nu_{k+1}+\nu_{k+2}}\} \\
&\leq \{(B_k+\gamma\mu^{\nu_k})(B_{k-1}+\gamma\mu^{\nu_{k-1}}) - \gamma^2\mu^{\nu_{k+1}}\}\mu^{\nu_{k+2}} \\
&\leq \{(B_k+\gamma\mu^{\nu_k})(B_{k-1}+\gamma\mu^{\nu_{k-1}}) - \gamma\mu^{\nu_{k+1}}\}\mu^{\nu_{k+2}} = B_{k+1}\mu^{\nu_{k+2}}. \qquad \square
\end{aligned}$$

5 φ with $|b-c|<1$

A sequence $\{f_k\}$ of functions with values in $\mathbb{R}\cup\{-\infty\}$ is called uniformly convergent on a set S when the sequence of non negative real valued functions $\{\exp(f_k)\}$ converges on S.

Theorem 5.1 *Assume $|b-c|<1$. The limit in (2.2) for φ of (1.1) converges uniformly on every compact in $\mathbb{C}^3\setminus\{0\}$. The limit function $G(p)$ satisfies the following properties:*
(1) *$G:\mathbb{C}^3\setminus\{0\}\to\mathbb{R}\cup\{-\infty\}$ is plurisubharmonic.*
(2) $G(\lambda p) = G(p) + \log|\lambda| \quad (\lambda\in\mathbb{C}^*)$.
(3) $G(\Phi(p)) = \omega G(p) + \frac{1}{\omega^2}\log|t|$.

I do not know whether the assumption $|b-c|<1$ is necessary for the convergence of the limit in (2.2). In the remaining part, we will prove Theorem 5.1.

In general, the multipliers of a holomorphic map h around a fixed point $P\in\mathbb{C}^2$ mean the two eigenvalues of the differential $dh(P)$. When absolute values of the two multipliers are both <1, the fixed point P is called attracting. For our maps φ or ψ^2, we will consider the blowing up of $\mathbb{P}^2$ at a point of indeterminacy. Then, a fixed point appears on the exceptional set which is attracting under some conditions on the parameters (b,c).
Using the notation in (1.6), define the coordinates $(u=\frac{z}{w}, v=\frac{t}{w})$ around I_1 such that $\mathbb{C}^2(u,v)=\mathbb{P}^2-\{w=0\}$ and $(f=\frac{w}{z}, g=\frac{t}{z})$ around I_2 such that $\mathbb{C}^2(f,g)=\mathbb{P}^2-\{z=0\}$. Let $\pi: M\to\mathbb{P}^2$ be the blowing up centered at the point I_1 with the exceptional curve $E=\pi^{-1}(I_1)$. Let us denote by L_w, L_z, L_t the proper transforms of $\{w=0\},\{z=0\},\{t=0\}\subset\mathbb{P}^2$, respectively. Let us denote by U_1,U_2,U_3 the open subsets of M which are biholomorphic to $\mathbb{C}^2$ defined by $U_1=M\setminus\{L_z\cup L_w\}, U_2=M\setminus\{L_w\cup L_t\}, U_3=M\setminus\{E\cup L_z\}$. Take the coordinates (r,s) of U_1 defined by $r=(\frac{z}{w})\circ\pi$, $s=(\frac{t}{z})\circ\pi$, (p,q) of U_2 by $p=(\frac{t}{w})\circ\pi$, $q=(\frac{z}{t})\circ\pi$. Since $\mathbb{C}^2(f,g)$ and U_3 are biholomorphic, we use the coordinates (f,g) of U_3 by abusing $f=f\circ\pi$ and $g=g\circ\pi$.
Denote three points of M by $\tilde{I}_1=\{(r,s)=(0,0)\}$, $\tilde{I}_2=\{(f,g)=(0,0)\}$ and $X=\{(p,q)=(0,-c)\}$. Let us denote by $\tilde{\varphi},\ \tilde{\psi}: M\cdots\to M$ the lift of $\varphi,\psi:\mathbb{P}^2\cdots\to\mathbb{P}^2$. Let us denote by $I(\varphi)$ the points of indeterminacy of φ.

Proposition 5.2 (1) *About the points of indeterminacy, it hold* $I(\varphi) = \{I_1, I_2\}$, $I(\tilde{\varphi}) = \{\tilde{I}_2\}$ *and* $I(\tilde{\varphi}^2) = \{X, \tilde{I}_2\}$.
(2) *We have* $\tilde{\varphi}(E) = L_t$, $\tilde{\varphi}(\tilde{I}_1) = \tilde{I}_1$, $\tilde{\varphi}(X) = \tilde{I}_2$, $\tilde{\varphi}(L_t \backslash \{\tilde{I}_2\}) = \tilde{I}_1$ *and* $\tilde{\varphi}^2(E\backslash\{X\}) = \tilde{I}_1$.
(3) *At the fixed point* $\tilde{I}_1$ *of* $\tilde{\varphi}$, *the multipliers are* $\{0, 0\}$.

Proof. By (1.1), we see $I(\varphi) = \{I_1, I_2\}$. Since $E \cap U_1 = \{r = 0\}$ and $E \cap U_2 = \{p = 0\}$, all the other assertions are verified from

$$\tilde{\varphi} : r_1 = \frac{s}{1 + brs + cs} = \frac{1}{q + bpq + c}, \; s_1 = rs = p; \; f_1 = q + bpq + c, \; g_1 = p;$$

and

$$\tilde{\varphi}^2 : r_2 = \frac{(1 + brs + cs)rs}{(1 + crs)(1 + brs + cs) + brs^2} = \frac{(q + bpq + c)p}{(1 + cp)(q + bpq + c) + bp}, \; s_2 = \frac{rs^2}{1 + brs + cs} = \frac{p}{q + bpq + c}.$$

Denote $J_2 = [-b : -bc : 1]$ and $J_3 = [-bc : b(bc - b - c^2) : 1]$. Then, $J_2 = J_3$ iff $c = 1$ or $b = 0$. For $\pi : M \to \mathbb{P}^2$, we set $\tilde{J}_2 = \pi^{-1}(J_2)$ and $\tilde{J}_3 = \pi^{-1}(J_3)$.

Proposition 5.3 (1) *We have* $I(\psi) = \{I_1, J_2\}$, $I(\tilde{\psi}) = \{\tilde{I}_1, \tilde{J}_2\}$, $I(\psi^2) = \{I_1, J_2, J_3\}$ *and* $I(\tilde{\psi}) = \{\tilde{I}_1, \tilde{J}_2, \tilde{J}_3\}$, *where* $J_2 = J_3$ *and* $\tilde{J}_2 = \tilde{J}_3$ *iff* $c = 1$ *or* $b = 0$.
(2) *Though* $\psi(I_2) = I_1 \in I(\psi)$, ψ^2 *is holomorphic at* I_2 *and* $\psi^2(I_2) = I_2$. *We have* $\tilde{\psi}(L_t\backslash\{\tilde{I}_1\}) = E\backslash\{\tilde{I}_1\}$, $\tilde{\psi}(E\backslash\{\tilde{I}_1\}) = \tilde{I}_2$ *and* $\tilde{\psi}(\tilde{I}_2) = X$. *Two points* $\{\tilde{I}_2, X\}$ *form a cycle of* $\tilde{\psi}$.
(3) *The multipliers at the fixed points* $\tilde{I}_2$ *and* X *of* $\tilde{\psi}^2$ *are both* $\{0, b - c\}$.

Proof. By (1.2), we see $I(\psi) = \{I_1, J_2\}$. Using the notation $(\hat{z}_k, \hat{w}_k, \hat{t}_k) = \frac{\Psi(p)}{\Delta_k(p)}$, we have

$$\hat{z}_2 = \{z^2 + (b + c^2)zt - cwt\}(z + bt), \; \hat{w}_2 = \{w + (b - c)z + b^2t\}(w - cz)t,$$

$$\hat{t}_2 = \{w + (b - c)z + b^2t\}(z + bt)t,$$

hence, we have the results about $I(\psi^2)$. The set $I(\tilde{\psi})$ and $I(\tilde{\psi}^2)$ can be seen from

$$\tilde{\psi} : r_1 = \frac{(1 - cr)s}{(1 + bs)r} = \frac{1 - cpq}{(q + b)pq}, \; s_1 = \frac{(1 + bs)r}{1 - cr} = \frac{(q + b)p}{1 - cpq};$$

$$\tilde{\psi} : f_1 = \frac{(q + b)pq}{1 - cpq}, \; g_1 = \frac{(q + b)p}{1 - cpq};$$

$$\tilde{\psi}^2 : p_2 = \frac{\{(q + c) + b - c\}p}{1 - cp(q + c) + c^2p}, q_2 + c = \frac{\{q^2 + (b + bc)q + cb^2\}p}{1 + (b - c)pq + b^2p}; \tag{5.1}$$

$$\tilde{\psi}^2 : r_2 = \frac{\{r + (b + c^2)rs - cs\}(1 + bs)r}{\{1 + (b - c)r + b^2rs\}(1 - cr)s}, \; s_2 = \frac{\{1 + (b - c)r + b^2rs\}s}{r + (b + c^2)rs - cs}.$$

The first assertion of (2) can be seen from $\psi : u_1 = \frac{(f-c)g}{1+bg}$, $v_1 = g$ and

$$\psi^2 : f_2 = \frac{\{f + (b - c) + b^2g\}(f - c)g}{\{1 + (b + c^2)g - cfg\}(1 + bg)}, \; g_2 = \frac{\{f + (b - c) + b^2g\}g}{1 + (b + c^2)g - cfg}, \tag{5.2}$$

and the other assertions of (2) from the above expression of $\tilde{\psi}$ and $\tilde{\psi}: p_1 = g,\ q_1 = \frac{(f-c)}{(1+bg)}$. Finally, the multipliers of $\tilde{\psi}^2$ at X and $\tilde{I}_2$ are seen from (5.1) and (5.2). $\square$

We will study around the attracting fixed point X and $\tilde{I}_2$ of $\tilde{\psi}^2$.

Lemma 5.4 *Assume* $|b-c|<1$.
(1) *We can take* $\eta_1 > 0$ *sufficiently small such that, for any sufficiently small* $\delta_1 > 0$*, letting* $T_1 = \{|p| < \delta_1, |q+c| < \eta_1\}$*, we have* $\tilde{\psi}^2(T_1) \subset T_1$.
(2) *We can take* $\delta_2 > 0$ *sufficiently small such that, for any sufficiently small* $\eta_2 > 0$*, letting* $T_2 = \{|f| < \delta_2, |g| < \eta_2\}$*, we have* $\tilde{\psi}^2(T_2) \subset T_2$.

Proof. (1) Fix $\eta_1 > 0$ with $|b-c| + \eta_1 < \frac{1+|b-c|}{2}$. Take any $\delta_1 > 0$ so far as it satisfies

$$\frac{|b-c|+\eta_1}{1-|c|\delta_1\eta_1 - |c|^2\delta_1} < \frac{1+|b-c|}{2} \text{ and } \frac{\delta_1\{(\eta_1+|c|)^2 + |b+bc|(\eta_1+|c|) + |c||b|^2\}}{1-|b-c|\delta_1(\eta_1+|c|) - |b|^2\delta_1} < \frac{\eta_1}{2}.$$

Then, according to (5.1), we can show that $\tilde{\psi}^2(T_1(\delta_1,\eta_1)) \subset T_1(\frac{1+|b-c|}{2}\delta_1, \frac{\eta_1}{2})$.
(2) Take and fix δ_2 with $0 < |b-c| + \delta_2 < \frac{1+|b-c|}{2}$. Let $\eta_2 > 0$ be any number with

$$\frac{\delta_2 + |b-c| + |b|^2\eta_2}{1-|b+c^2|\eta_2 - |c|\delta_2\eta_2} < \frac{1+|b-c|}{2} \text{ and } \frac{(\delta_2+|b-c|+|b|^2\eta_2)(\delta_2+|c|)\eta_2}{(1-|b+c^2|\eta_2 - |c|\delta_2\eta_2)(1-|b|\eta_2)} < \frac{\delta_2}{2}.$$

Then, according to (5.2), we can show that $\tilde{\psi}^2(T_2(\delta_2,\eta_2)) \subset T_2(\frac{\delta_2}{2}, \frac{1+|b-c|}{2}\eta_2)$. $\square$

Let $\mathbb{C}^2 = \mathbb{P}^2 \backslash \{t=0\}$ with the coordinates $x = \frac{z}{t}, y = \frac{w}{t}$. Using $\eta_1, \delta_2, \eta_2, \delta_1 > 0$ satisfying $|c| + \eta_1 < \frac{1}{\eta_2}$, define

$$V_1 = \overline{\{|y| \geq 1/\delta_1,\ |x| \leq 1/\eta_2\} - \{|y| \geq 1/\delta_1,\ |x+c| \leq \eta_1\}} \cup \{\delta_2|x| \leq |y|,\ |x| \geq 1/\eta_2\},$$

$$V_2 = \{|y| \leq \delta_2|x|,\ |x| \geq 1/\eta_2\}, \quad V_3 = \{|y| \geq \frac{1}{\delta_1},\ |x+c| \leq \eta_1\},$$

so that $\mathbb{C}^2 \backslash (V_1 \cup V_2 \cup V_3) = \{|x| < \frac{1}{\eta_2},\ |y| < \frac{1}{\delta_1}\}$. Let $\Omega_1, \Omega_2, \Omega_3 \subset M$ be the attracting basins of $\tilde{\varphi}$ at $\tilde{I}_1$, $\tilde{\psi}^2$ at $\tilde{I}_2$ and $\tilde{\psi}^2$ at X, respectively.

Proposition 5.5 *Assume* $|b-c|<1$. *We can choose* $\eta_1,\ \delta_2,\ \eta_2,\ \delta_1 > 0$ *so that Lemma (5.4) holds and furthermore we have* $V_1 \subset \pi(\Omega_1) \cap \mathbb{C}^2$, $V_2 \subset \pi(\Omega_2) \cap \mathbb{C}^2$ *and* $V_3 \subset \pi(\Omega_3) \cap \mathbb{C}^2$.

Proof. We will define $V_1' \subset (U_1 \cup U_2) \subset M$ by

$$V_1' = \overline{\{|p| \leq \delta_1,\ |q| \leq 1/\eta_2\} - \{|p| \leq \delta_1,\ |q+c| \leq \eta_1\}} \cup \{|r| \leq 1/\delta_2,\ |s| \leq \eta_2\},$$

$V_2' = \{|f| \leq \delta_2,\ |g| \leq \eta_2\} \subset \mathbb{C}^2(f,g) \subset \mathbb{P}^2$, and $V_3' = \{|p| \leq \delta_1,\ |q+c| \leq \eta_1\} \subset U_2 \subset M$. First we choose η_1 and δ_2 sufficiently small. Secondly, in view of Proposition 5.2 (2), we choose η_2 sufficiently small so that Lemma (5.4) (2) and $\{|r| \leq \frac{1}{\delta_2},\ |s| \leq \eta_2\} \subset \Omega_1$ hold.

Finally, in view of Lemma (5.4) (1) and Proposition 5.2 (2), we can take δ_1 so that $V_1' \subset \Omega_1$ hold. □

As an analogy of the theory of the Hénon map ([BS]), let

$$K_+ = \{P \in \mathbb{C}^2; \{\varphi^k(P); k\} \text{ is bounded in } \mathbb{C}^2\}.$$

Proposition 5.6 *Assume $|b-c| < 1$. For $A > 0$ with (4.1), it holds $\bigcup_{k=1}^{\infty} \varphi^{-k}(W_A) = \mathbb{C}^2 \backslash K_+$.*

Proof. By Lemma 4.1, it is clear that $\bigcup_{k=1}^{\infty} \varphi^{-k}(W_A) \subset \mathbb{C}^2 \backslash K_+$. Conversely, let $P \in \mathbb{C}^2 \backslash K_+$ be a point, and assume that $P \notin \bigcup_{k=1}^{\infty} \varphi^{-k}(W_A)$. Consider the configuration as in Proposition 5.5. At this moment, we can assume that the point of indeterminacy J_2 of ψ belongs to $\{|x| < \frac{1}{\eta_2}, |y| < \frac{1}{\delta_1}\}$. By Lemma 5.4, we can take $m_0 \in \mathbb{N}$ such that

$$P \notin \psi^m(V_2 \cup V_3) \text{ for all } m > m_0. \tag{5.3}$$

Since the set $\{\varphi^k(P); k\} \subset (\mathbb{C}^2 \setminus V_1)$ is unbounded, Proposition 5.5 assures the existence of $k > m_0$ such that $\varphi^k(P) \in (V_2 \cup V_3)$. Then we have $P = \psi^k(\varphi^k(P)) \in \psi^k(V_2 \cup V_3)$, which is a contradiction to (5.3).

Lemma 5.7 *Assume $|b-c| < 1$. Let L be a compact such that $L \cap K_+ \neq \emptyset$. Then $\{g_k(q)\}$ converges to $g(q)$ uniformly on L.*

Proof. Take A with (4.1). Next take $\eta_1, \delta_2, \eta_2, \delta_1 > 0$ satisfying the configuration of Proposition 5.5 and the conditions $A < \frac{1}{\eta_2}$, $A < \frac{A(1+|b|)}{\eta_1} < \frac{1}{\delta_1}$ and $L \subset \{|x| < A,\ |y| < \frac{1}{\delta_1}\}$. Set $F = \{|x| \le A,\ |y| \le \frac{1}{\delta_1}\}$. Let $S = F \cup \varphi(F) \cup \varphi^2(F) \cup \varphi^3(F)$ and let $T = \max_{(x,y)\in S}(\log|x|, \log|y|, 0)$. Since $W_A \cap S$ is compact, Poposition 4.5 enables us to take $M > 0$ such that

$$g_k \le M \text{ on } W_A \cap S \text{ for any } k \in \mathbb{N}. \tag{5.4}$$

Let us take any $\varepsilon > 0$ and any $q \in L$. We will use the notation $q_k = (x_k, y_k) = \varphi^k(q)$. Take $N > 0$ such that $M \frac{\nu_{k+2}}{\nu_{k+N+2}} < \frac{\varepsilon}{2}$ for all $k \in \mathbb{N}$. Take $K_2 > 0$ such that

$$\frac{1}{\nu_{k+2}} T < \frac{1}{\varepsilon} \text{ for all } k \ge K_2. \tag{5.5}$$

First, when $q \in L \cap K_+$, since Proposition 5.5 guarantees $q_k \in F$ for all $k \in \mathbb{N}$, we have

$$0 \le g_k(q) = \frac{1}{\nu_{k+2}} \max(\log|x_k|, \log|y_k|, 0) < \frac{\varepsilon}{2} \text{ for all } k \ge K_2. \tag{5.6}$$

Secondly, Proposition 4.5 assures the existence of $K_3 > 0$ such that

$$|g_k - g_l| < \varepsilon \text{ on } L \cap \bigcup_{k=1}^{N} \varphi^{-k}(W_A) \text{ and for all } k > l \ge K_3. \tag{5.7}$$

Finally, it remains the case where $q \in L \cap (\mathbb{C}^2 \setminus \bigcup_{k=1}^{N} \varphi^{-k}(W_A))$. We remark that Poposition 5.5 assures $\varphi^k(q) \notin (V_2 \cup V_3)$ for all $k \in \mathbb{N}$. We will prove that $g_k(q) < \frac{\varepsilon}{2}$ for $k > K_2$, hence

$$|g_k(q) - g_l(q)| < \varepsilon \text{ for } k > l > K_2. \tag{5.8}$$

Let $k_1 > N$ be the smallest integer with $q_{k_1} \in W_A$. We will show that

$$q_k \in F \text{ or } q_{k+1} \in F \text{ for } 1 \leq k \leq k_1 - 3. \tag{5.9}$$

Let $1 \leq k \leq k_1 - 3$ and assume $q_k \notin F$. If q_k satisfies $|x_k + c| \geq \eta_1, |x_k| \leq A$ and $|y_k| \geq \frac{1}{\delta_1}$, then we have $|x_{k+1}| = |y_k| \geq A$ and $|y_{k+1}| \geq |x_k + c||y_k| - |b||x_k| \geq \eta_1 |y_k| - |b|A \geq A$, which implies $q_{k+1} \in W_A$, a contradiction to the definition of k_1. So, $|x_k| \geq A$. Since $q_k \notin W_A$, it follows $|y_k| \leq A$, hence $|x_{k+1}| = |y_k| \leq A$. If $|x_{k+1} + c| \geq \eta_1, |y_{k+1}| \geq \frac{1}{\delta_1}$, then the same argument reaches $q_{k+2} \in W_A$, which is a contradiction. Hence, $q_{k+1} \in F$.
Now we will show that $g_{k+1}(q) \leq \frac{\varepsilon}{2}$ for all k with $K_2 \leq k \leq k_1 - 3$. In fact, because of (5.5), $g_{k+1}(q) = \frac{1}{\nu_{k+3}} \log^+ |\varphi(q_k)| \leq \frac{\varepsilon}{2}$ if $q_k \in F$, and $g_{k+1}(q) = \frac{1}{\nu_{k+3}} \log^+ |q_{k+1}| \leq \frac{\varepsilon}{2}$ if $q_{k+1} \in F$. Rephrasing this statement, $g_k(q) \leq \frac{\varepsilon}{2}$ when $K_2 + 1 \leq k \leq k_1 - 2$.
Because of (5.9), for k with $K_2 \leq k_1 - 1 \leq k \leq k_1$, $q_k \in (\varphi(F) \cup \varphi^2(F) \cup \varphi^3(F))$. So, (5.5) yields $g_k(q) = \frac{1}{\nu_{k+2}} \log |q_k| \leq \frac{1}{\nu_{k+2}} T \leq \frac{\varepsilon}{2}$.
Let $k \geq k_1 + 1$. Then $g_k(q) = g_{k-k_1}(q_{k_1}) \frac{\nu_{k-k_1+2}}{\nu_{k+2}} \leq M \frac{\nu_{k-N+2}}{\nu_{k+2}} < \frac{\varepsilon}{2}$.
Gathering three cases, we have proved (5.8). Now the assertions (5.6), (5.7), (5.8) complete the proof of the Proposition. □
Proof of Theorem 5.1 We will concentrate on the proof of the convergence of (2.2). After proving the convergence, we have (1) and (2) clearly. In view of (4.9), we have

$$G_k(\Phi(p)) = \frac{1}{\nu_{k+2}} \log |\Phi(\hat{z}_k, \hat{w}_k, \hat{t}_k)| = \frac{\nu_{k+3}}{\nu_{k+2}} \frac{1}{\nu_{k+3}} \log \max(|\hat{z}_{k+1}|, |\hat{w}_{k+1}|, |\hat{t}_{k+1}|) + \frac{\nu_k}{\nu_{k+2}} \log |t|,$$

hence (3) follows by letting $k \to \infty$.
First Step $\{G_k(p)\}$ converges uniformly on every compact of of $\rho^{-1}(\mathbb{P}^2 \setminus \{t = 0\})$.
By Propositions 4.5 and 5.6, $\{g_k\}$ converges uniformly on every compact of $\mathbb{C}^2 \setminus K^+$. So, Lemma 5.7 guarantees that $\{g_k\}$ converges uniformly on every compact in $\mathbb{C}^2$. Since $G_k(z, w, t) = G_k(\frac{z}{t}, \frac{w}{t}, 1) + \log |t| = g_k(\frac{z}{t}, \frac{w}{t}) + \log |t|$ on $\rho^{-1}(\mathbb{C}^2)$, the first step is completed.
Second Step $\{G_k(p)\}$ converges uniformly on every compact in $\rho^{-1}(\mathbb{P}^2 \setminus \{I_1, I_2\})$.
As a consequence of the first step, it suffices to prove the uniform convergence of $\{G_k(p)\}$ on the set $D(\alpha, \beta) = \{\alpha \leq |z| \leq 1/\alpha,\ \alpha \leq |w| \leq 1/\alpha,\ |t| \leq \beta\}$, where $0 < \alpha < 1$ and $0 < \beta < 1$. Let us take any $\varepsilon > 0$. Choose $0 < \delta < 1$ such that

$$\left(\frac{\nu_{k+2} - 1}{\nu_{k+2}} + \frac{\nu_{l+2} - 1}{\nu_{l+2}}\right) \log(1 + \delta) < \varepsilon \text{ for all } k, l \in \mathbb{N}. \tag{5.10}$$

Let us take $A_1 > 0$ of (4.2) for this δ. Then estimates of Lemma 4.3 hold. By the notation of Propositions 4.5 and 2.4, since $x_k = \frac{\hat{z}_k}{\hat{t}_k}$, $y_k = \frac{\hat{w}_k}{\hat{t}_k}$, we have

$$(1 + \delta)^{-(\nu_{k+1} - 1)} |z|^{\nu_{k-1}} |w|^{\nu_k} |t|^{\nu_k} \leq |\hat{z}_k| \leq (1 + \delta)^{(\nu_{k+1} - 1)} |z|^{\nu_{k-1}} |w|^{\nu_k} |t|^{\nu_k},$$

$$(1+\delta)^{-(\nu_{k+2}-1)}|z|^{\nu_k}|w|^{\nu_{k+1}} \le |\hat{w}_k| \le (1+\delta)^{(\nu_{k+2}-1)}|z|^{\nu_k}|w|^{\nu_{k+1}}$$

on $\{A_1|t| \le |z|,\ A_1|t| \le |w|\}$. Then, we can choose $0 < \beta_1 < \beta$ such that $G_k(z,w,t) = \frac{1}{\nu_{k+2}} \log|\hat{w}_k|$ when $(z,w,t) \in D(\alpha,\beta_1)$. Let us take $K_1 > 0$ such that

$$|\frac{\nu_k}{\nu_{k+2}} - \frac{\nu_l}{\nu_{l+2}}| \log\frac{1}{\alpha} < \varepsilon,\ |\frac{\nu_{k+1}}{\nu_{k+2}} - \frac{\nu_{l+1}}{\nu_{l+2}}| \log\frac{1}{\alpha} < \varepsilon \text{ for all } k > l \ge K_1. \tag{5.11}$$

Then we see that

$$|G_k(p) - G_l(p)| \le 3\varepsilon \text{ for } k > l \ge K_1 \text{ and } p \in D(\alpha,\beta_1). \tag{5.12}$$

On the other hand, since $D(\alpha,\beta)\backslash D(\alpha,\beta_1) \subset\subset \mathbb{C}^3\backslash\{t=0\}$, First Step assures the existence of K_2 such that

$$|G_k(p) - G_l(p)| \le \varepsilon \text{ for } k > l \ge K_2 \text{ and } p \in D(\alpha,\beta) \setminus D(\alpha,\beta_1). \tag{5.13}$$

Now, (5.12) and (5.13) implies the uniform convergence of G_k on $D(\alpha,\beta)$, which completes the proof od Second Step.

Final Step It remains to prove that $\{G_k(p)\}$ converges uniformly on $\rho^{-1}(W_1 \cup W_2) \cap L$, where W_i is a neighborhood of the point I_i in $\mathbb{P}^2$ and L is a compact of $\mathbb{C}^3\backslash\{0\}$. Let us take an arbitrary $\varepsilon > 0$. Then, by Proposition 4.6, there exists $\mu > 0$ and $K_1 > 0$ such that,

$$\exp(|G_k(p)|) < \frac{\varepsilon}{2} \text{ for } p \in \rho^{-1}(\Lambda_1(\mu) \cup \Lambda_2(\mu)) \cap L \text{ and for } k \ge K_1. \tag{5.14}$$

On the other hand, Second Step assures the existence of $K_2 \ge K_1$ such that, for all $k > l \ge K_2$,

$$|G_k - G_l| < \varepsilon \text{ on } \rho^{-1}((W_1\backslash\Lambda_1(\mu)) \cup (W_2\backslash\Lambda_2(\mu))) \cap L \text{ in } \rho^{-1}(\mathbb{P}^2\backslash\{I_1,I_2\}). \tag{5.15}$$

Now, (5.14) and (5.15) implies the assertion of Final Step, which completes the proof of Theorem 5.1.

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